

# A Mass Formula for EYM Solitons

Alejandro Corichi<sup>1\*</sup>, Ulises Nucamendi<sup>2</sup> and Daniel Sudarsky<sup>1</sup>

*1. Instituto de Ciencias Nucleares*

*Universidad Nacional Autónoma de México*

*A. Postal 70-543, México D.F. 04510, México.*

*2. Center for Theoretical Physics*

*University of Sussex*

*Brighton BN1 9QJ, Great Britain.*

## Abstract

The Isolated Horizon formalism, together with a simple phenomenological model for colored black holes was recently used to predict a formula for the ADM mass of the solitons of the EYM system in terms of horizon properties of black holes *for all* values of the horizon area. In this note, this formula is tested numerically –up to a large value of the area– for spherically symmetric solutions and shown to yield the known masses of the solitons.

04.70.Bw, 04.20.Jb

Typeset using REVTeX

---

\*E-mail address: corichi@nuclecu.unam.mx

Historically, the use of simple models has played an important role to gain intuition about the behavior of complex gravitational systems. For example, the intuitive notion of “what can be radiated *will* be radiated” after a gravitational collapse gave way to the formulation of the “no-hair conjecture” for stationary solutions [1]. Similarly, the fact that there exist no solitonic (i.e. stationary, regular) solutions to the Yang-Mills equations on Minkowski space-time was in part responsible for the general belief that solitons were also absent in the presence of gravity. However, as these examples illustrate, the intuitive models that one constructs are of a limited validity once one considers more general situations. For instance, it is now known that the Einstein-Yang-Mills system violates both of the early “conclusions”, that is, there exist both solitons [2,3] and colored black holes [4,5] in that theory. More recently, the use of dimensional analysis has helped to develop intuition about the existence of static (regular and BH type) solutions in theories with different matter content [6]. The basic idea is that whenever a dimension-full constant can be constructed from the fundamental coupling constants of the theory, then there exist non-trivial static solutions. For example, in the EYM system the gravitational constant  $G$  and the gauge coupling constant  $g$  give rise to a mass parameter (or a length scale) that governs the scale of the mentioned solutions. One of the facts that can be heuristically explained from this viewpoint is the emergence of new phenomena once more fields (and thus more coupling constants) are introduced. For instance, in the passage from EYM to EYM-Higgs, a new mass parameter can be constructed from the expectation value  $\eta$  of the Higgs field. This indicates, for instance, that the hair of the hairy black holes is always short and that those BH have an upper bound on their area [7].

During the past year, the introduction of the Isolated Horizon (IH) formalism [9,10] has allowed to gain insight into the static sector of theories admitting “hair” [11]. First, it was found that the Horizon Mass of the black hole, constructed purely of quasi-local quantities, is related in a simple way to the ADM masses of both colored black holes and solitons of the theory [11]. Second, a simple model of colored black holes as bound states of regular black holes and solitons emerged and provided heuristic explanations for the behavior of horizon quantities of colored black holes [12]. Finally, the combination of the Mass formula, together with the fact that in EYMH theory, different “branches” of static solutions merge, has lead to a formula for the difference of soliton masses in terms of black hole quantities [13,12]. In this article, we explore further the consequences of the IH formalism in the static sector of the theory. In particular, we explore a formula for the ADM mass of the  $n$  Bartnik-McKinnon soliton [2] in terms of horizon quantities of the related black holes,

$$M_{\text{sol}}^{(n)} = \frac{1}{2G} \int_0^\infty (1 - \beta_{(n)}(r)) dr \quad (1)$$

where  $\beta_{(n)}(r) = 2r\kappa_{(n)}(r)$  and  $\kappa_{(n)}(r)$  is the surface gravity of the  $n$  colored black hole of area  $a_\Delta = 4\pi r^2$ . This formula is striking because it allows to compute masses of *solitons* from surface gravity of *black holes*. It should be noted that even when this formula refers to the EYM system, it was predicted from considerations in the EYMH system [12].

Two basic ingredients are needed for the validity of this formula: i) That *the static sector* of the EYMH system, in the limit when the vacuum expectation value  $\eta$  approaches zero ( $\eta \mapsto 0$ ) reduces to the static sector of the EYM system. Thus, the solutions of the EYM system would correspond to limiting solutions of EYMH equations; ii) The surface gravity

of colored black holes should have certain limiting behavior as the horizon area grows (in order to have convergence of the integral in (1)). In this note we shall prove i) and, by numerical integration, we show evidence for the mass formula (1), and thus indirectly of the limiting behavior of surface gravity. Our numerical integration is restricted to the spherically symmetric sector and for the  $n = 1, 2$  colored black holes, but one expects the main result to be valid in general, non-spherical solutions and for all values of  $n$ .

Let us begin by recalling the *Isolated Horizons* (IH) formalism, whose applications range from the extraction of physical quantities in numerical relativity to quantum entropy calculations [8]. The basic idea is to consider space-times with an interior boundary (to represent the horizon), satisfying quasi-local boundary conditions ensuring that the horizon remains ‘isolated’. Although the boundary conditions are motivated by geometric considerations, they lead to a well defined action principle and Hamiltonian framework. Furthermore, the boundary conditions imply that certain ‘quasi-local charges’, defined at the horizon, remain constant ‘in time’, and can thus be regarded as the analogous of the global charges defined at infinity in the asymptotically flat context. The isolated horizon Hamiltonian framework enables one to define the *Horizon Mass*  $M_\Delta$ , as function of the ‘horizon charges’.

In the Einstein-Maxwell and Einstein-Maxwell-Dilaton systems considered originally [9], the horizon mass satisfies a Smarr-type formula and a generalized first law in terms of quantities defined exclusively at the horizon (i.e. without any reference to infinity). The introduction of non-linear matter fields like the Yang-Mills field brings unexpected subtleties to the formalism [11]. However, one can still define a Horizon Mass, and furthermore, this Horizon Mass satisfies a first law.

An isolated horizon is a non-expanding null surface generated by a (null) vector field  $l^a$ . The IH boundary conditions imply that the acceleration  $\kappa$  of  $l^a$  ( $l^a \nabla_a l^b = \kappa l^b$ ) is constant on the horizon  $\Delta$ . However, the precise value it takes on each point of phase space is not determined a priori. On the other hand, it is known that for each vector field  $t_o^a$  on space-time, the induced vector field  $X_{t_o}$  on phase space is Hamiltonian if and only if there exists a function  $E_{t_o}$  such that  $\delta E_{t_o} = \Omega(\delta, X_{t_o})$ , for any vector field  $\delta$  on phase space. This condition can be re-written as [10],  $\delta E_{t_o} = \frac{\kappa_{t_o}}{8\pi G} \delta a_\Delta + \text{work terms}$ . Thus, the first law arises as a necessary and sufficient condition for the consistency of the Hamiltonian formulation; the allowed vector fields  $t^a$  will be those for which the first law holds. Note that there are as many ‘first laws’ as allowed vector fields  $l^a \triangleq t^a$  on the horizon. However, one would like to have a *physical first law*, where the Hamiltonian  $E_{t_o}$  be identified with the ‘physical mass’  $M_\Delta$  of the horizon. This amounts to finding the ‘right  $\kappa$ ’. This ‘normalization problem’ can be easily overcome in the EM system [9]. In this case, one chooses the function  $\kappa = \kappa(a_\Delta, Q_\Delta)$  as the corresponding function for the *static* solution with charges  $(a_\Delta, Q_\Delta)$ . However, for the EYM system, this procedure is not as straightforward. A consistent viewpoint is to abandon the notion of a globally defined horizon mass on Phase Space, and to define, for each value of  $n = n_o$ , a canonical normalization  $t_{n_o}^a$  that yields the Horizon Mass  $M_\Delta^{(n_o)}$  for the  $n_o$  branch [10]. The horizon mass takes the form,

$$M_\Delta^{(n_o)}(R_\Delta) = \frac{1}{2G} \int_0^{R_\Delta} \beta_{(n_o)}(r) dr, \quad (2)$$

along the  $n_o$  branch with  $R_\Delta$  the horizon radius.

Finally, for any  $n$  one can relate the horizon mass  $M_{\Delta}^{(n)}$  to the ADM mass of static black holes. Recall first that general Hamiltonian considerations imply that the total Hamiltonian, consisting of a term at infinity, the ADM mass, and a term at the horizon, the Horizon Mass, is constant on every connected component of static solutions (provided the evolution vector field  $t^a$  agrees with the static Killing field everywhere on this connected component) [9,10]. In the Einstein-Yang-Mills case, since the Hamiltonian is constant on any  $n$ -branch, we can evaluate it at the solution with zero horizon area. This is just the soliton, for which the horizon area  $a_{\Delta}$ , and the horizon mass  $M_{\Delta}$  vanish. Hence we have that  $H^{(t_0,n)} = M_{\text{sol}}^{(n)}$ . Thus, we conclude:

$$M_{\text{sol}}^{(n)} = M_{\text{ADM}}^{(n)} - M_{\Delta}^{(n)} \quad (3)$$

on the entire  $n$ th branch [11,10]. Thus, the ADM mass contains two contributions, one attributed to the black hole horizon and the other to the outside ‘hair’, captured by the ‘solitonic residue’. The formula (3), together with some energetic considerations, lead to the model of a colored black hole as a bound state of an ordinary, ‘bare’, black hole and a ‘solitonic residue’, where the ADM mass of the colored black hole of radius  $R_{\Delta}$  is given by the ADM mass of the soliton plus the horizon mass of the ‘bare’ black hole plus the binding energy [12]. Simple considerations about the behavior of the ADM masses of the colored black holes and the solitons, together with some expectations of this model (such as a negative binding energy) give raise to prediction of the behavior of horizon parameters [12]. Among the predictions, we have that  $\beta_{(n)}(R_{\Delta})$ , as a function of  $R_{\Delta}$  and  $n$ , is a positive function, bounded above by  $\beta_{(0)}(R_{\Delta}) = 1$ . Besides, the curves  $\beta_{(n)}(r)$ , as functions of  $r$  intersect the  $r = 0$  axis at distinct points between 0 and 1, never intersect, and have the property that the higher  $n$  is, the lower the curve. Finally, the curves, for large value of their argument, become asymptotically tangential to the curve  $\beta_{(0)}(R_{\Delta}) = 1$ . One of the features of these solutions is that there is no limit for the size of the black hole. That is, if we plot the ADM mass of the BH as function of the radius  $R_{\Delta}$  we get an infinite number of curves, each of them intersecting the  $R_{\Delta} = 0$  line at the value of the soliton mass, and never intersecting each other.

When one introduces extra fields, as is the case of a Higgs field in EYM-Higgs, then there is an extra dimension-full parameter  $\eta$  that brings a new mass scale into the problem. This, in turn, has as a consequence that there is a maximum value for the horizon radius of the black holes [7,12]. This also indicates that in the  $M_{\text{ADM}}$  vs.  $R_{\Delta}$  plot, the curves corresponding to two families of black holes intersect at a point. Now, recall that the horizon Mass was obtained by integrating  $M_{\Delta}^{(n_o)}(R_{\Delta}) = \frac{1}{2G} \int_0^{R_{\Delta}} \beta_{(n_o)}(r) dr$ , along the curve corresponding to the  $n$ th family. If now two such curves, say the  $n$ -th and the  $n + 1$ -th, intersect, one can use this formula together with Eq. (3), to conclude that,

$$M_{\text{sol}}^{(n+1)} - M_{\text{sol}}^{(n)} = \frac{1}{2G} \oint \beta(r) dr \quad (4)$$

where the closed counter integral is performed by first moving along the  $n$ -th branch up to the crossing point and then returning along the  $n + 1$ -th branch back to  $R_{\Delta} = 0$  [13,12]. This concludes the summary of the IH formalism needed for this note.

Now let us consider the static sector of a family of EYMH theories, parametrized by  $\eta$ . As  $\eta$  decreases, the second mass parameter of the theory becomes smaller and the upper

bound on the area of the hairy black holes increases [7]. Thus we expect that the point of intersection of the  $n$  and  $n + 1$  branches would move towards larger values of  $R$  as  $\eta$  decreases. Moreover in the limit  $\eta \rightarrow 0$  we would expect this intersection point to move towards infinity leading to a situation where the different branches do not intersect, as is in fact observed to happen in pure EYM theory. We might also argue that as  $\eta$  determines the natural scale for the Higgs field in static situations, we expect that as  $\eta \rightarrow 0$  the static sector of EYMH theory would have vanishing Higgs field and thus correspond to the static sector of pure EYM theory. We will support this argument by explicitly proving that for the case  $\eta = 0$  the static, purely magnetic solutions of EYMH theory have vanishing Higgs field. The proof is a simple generalization of a Bekenstein no hair theorem [15].

Consider an EYMH theory with scalar field  $\Phi$  with potential  $V(\Phi) = \lambda(\Phi^*\Phi - \eta^2)^2$ , where  $\lambda$ , and  $\eta$  are constants, and  $\Phi^*$  is the Hermitian conjugate of  $\Phi$ . Consider a static black hole solution with time-like Killing field  $\xi^a$ . The equation of motion for the scalar field is:

$$D^a D_a \Phi - \frac{\partial V}{\partial \Phi^*} = 0 \quad (5)$$

where  $D_a$  stands for the gauge and metric covariant derivative  $D_a = \nabla_a - ieA_a^i T^i$ , with  $\nabla_a$  is the metric compatible derivative operator,  $A_a^i$  stand for the gauge fields,  $T^i$  for the Lie algebra generators, and  $e$  is the gauge coupling constant. For non-extremal black holes without loss of generality we can consider that the space-time has a bifurcate Killing Horizon with bifurcation surface  $S$  [14]. Let  $t$  be the Killing parameter and consider  $M$  the region of space-time bounded by  $\Sigma_1$  and  $\Sigma_2$  hyper-surfaces of constant  $t$ ,  $S$  and asymptotic infinity. We multiply Eq. (5) by  $\Phi^*$  and integrate over  $M$ :

$$\begin{aligned} 0 &= \int_M d^4x (\Phi^* D^a D_a \Phi - \Phi^* \frac{\partial V}{\partial \Phi^*}) \sqrt{-g} \\ &= \int_{\partial M} d\Sigma^a \Phi^* D_a \Phi - \int_M d^4x \sqrt{-g} (D_a \Phi^* D_b \Phi g^{ab} \\ &\quad + \Phi^* \frac{\partial V}{\partial \Phi^*}) \end{aligned} \quad (6)$$

Consider the case where  $\eta = 0$ . The boundary integral consists of four terms: The integrals over  $\Sigma_1$  and  $\Sigma_2$  that are equal in magnitude and opposite in sign due to the time translation invariance and the vanishing of the integral at infinity because of the fall-off conditions on  $\Phi$  required from asymptotic flatness. Finally, the term associated with  $S$  which does not contribute since the integral is over a lower dimensional manifold.

Next we write the inverse metric  $g^{ab} = -N^2 \xi^a \xi^b + h^{ab}$ , where  $N^{-1}$  is the norm of the Killing field  $\xi^a$  and  $h^{ab}$  is the inverse of spatial metric on  $\Sigma$ . Using the fact that  $\xi^a \nabla_a \Phi$  vanishes as long as the solution is static, and that  $A_0^i \equiv \xi^a A_a^i = 0$  corresponding to the purely magnetic sector, we obtain:

$$\int_M d^4x \sqrt{-g} (D_a \Phi^* D_b \Phi h^{ab} + \Phi^* \frac{\partial V}{\partial \Phi^*}) = 0 \quad (7)$$

The first term in the integrand is positive semi-definite in general and the second term becomes positive semi-definite when  $\eta = 0$ . Thus in this situation the only possibility is  $D_a \Phi = 0$  and  $\Phi^* \frac{\partial V}{\partial \Phi^*} = 0$  which for  $\eta = 0$  implies  $\Phi = 0$ . This is what we wanted to show.

TABLE I.  $R$  is the upper limit of the integration interval.  $I(n = 1)$ ,  $I(n = 2)$ ,  $I(n = 2) - I(n = 1)$  are the values of the integral for one node, two nodes, and the difference of both, respectively. The complete output of the simulation is reported even when the numerical errors make some of the last figures irrelevant.

R	$I(n = 1)$	$I(n = 2)$	$I(n = 2) - I(n = 1)$
1	0.39103324877641	0.47715590602640	8.6122657249990D-02
5	0.73246896963857	0.87112010238884	0.13865113275027
10	0.78027992488956	0.92106063175122	0.14078070686166
20	0.80423765124767	0.94603243120791	0.14179477996024
30	0.81222747739806	0.95435647857942	0.14212900118136
40	0.81622282131691	0.95851851479659	0.14229569347968
50	0.81862012691719	0.96101573942668	0.14239561250949
60	0.82021836327953	0.96268055681115	0.14246219353162
70	0.82135997388031	0.96386971245266	0.14250973857235
80	0.82221618794342	0.96476157934352	0.14254539140010
90	0.82288213542783	0.96545525369931	0.14257311827148
100	0.82341489514230	0.96601019325204	0.14259529810974
110	0.82385079057060	0.96601019325204	0.14261344416386
120	0.82421403734650	0.96684260265416	0.14262856530766
130	0.82452140046910	0.96716276014096	0.14264135967186
140	0.82478485490025	0.96743718084997	0.14265232594972
150	0.82501318223082	0.96767501213306	0.14266182990224

Thus, by considering the limit of the static and purely magnetic sector of EYMH theory in the limit  $\eta \rightarrow 0$  we expect to obtain the static and purely magnetic sector of the EYM theory. Furthermore, by considering Eq. (4) in this limit we obtain an equivalent equation (1), now for the EYM case. Let us now consider Minkowski space-time as the zeroth (degenerate) soliton, where the Schwarzschild branch (the 0<sup>th</sup> branch) begins, we obtain a formula for the mass of the  $n$ th soliton in EYM theory:

$$M_{\text{sol}}^{(n)} = \frac{1}{2G} \oint \beta(r) dr, \quad (8)$$

where the integral is performed first along the Schwarzschild branch to  $r = \infty$  and returning to  $r = 0$  along the  $n^{\text{th}}$  branch. This is precisely Eq. (1).

Now we want to establish the validity of formula (1) for EYM solitons. For that, we show a comparison between the numerical evaluation of the masses of the EYM solitons and the corresponding evaluation of the left hand side of Eq. (1) for the  $n = 1, 2$  colored black holes. To do this, we calculate numerically the functions  $\beta_{(n)}(r)$  for  $n = 1, 2$  as a function of the radius of the horizon  $R_{\Delta}$  by using a fourth-order Runge-Kutta algorithm for the computation of the configurations of black holes with  $n = 1, 2$  nodes maintaining the truncation error

below of the accuracy  $10^{-6}$  for the metric and the Yang-Mills field of each configuration. At the same time, a fifth-order Runge-Kutta algorithm was used for estimating the error in the values of the function  $\beta_{(n)}(r)$ . Then we calculate the value of the integral (1) for different values (sufficiently large) of the upper limit of integration  $R$  and the respective accumulated error associated with it coming of the one associated with the values of the function  $\beta_{(n)}(r)$ . The accumulated errors associated with the integral (1), calculated as the difference between the fourth and fifth order algorithms, for  $n = 1, 2$  and superior limit of integration  $R = 150$ , are  $5 \times 10^{-4}$  and  $9 \times 10^{-4}$  respectively (the accumulated errors for  $R > 150$  differ of these values by a quantity of the order  $10^{-10}$  and then have practically converged to these). The black hole radius is in the standard units given by the EYM coupling constant (where  $R = 1$  corresponds to the natural length scale given by the theory). Table I shows the values of the integral (1) for  $n = 1, 2$  and the difference between both as a function of the upper limit of integration  $R$ , using the fifth-order algorithm. It is important to notice from the Table that the difference of the integrals, which by equations (1) and (4) should approximate the difference in soliton masses, has indeed a faster convergence than the integrals by themselves. We have restricted the upper limit in the integral to  $R = 150$  given that the difference in masses had practically converged to the reported value.

From the values given in Table I, it is clear that, For  $R$  sufficiently large,, the integrals for  $n = 1, 2$  (and the difference of both) are approaching to the corresponding soliton masses, 0.82864698216 and 0.97134549426 for  $n = 1, 2$  respectively (and the difference of them, 0.142698512), as reported in [16].

To conclude, within the uncertainty of our numerical calculation, the soliton mass formula (1) provides the ADM mass of the EYM system.

We would like to thank A. Ashtekar for a careful reading of the manuscript and J. Smoller for pointing out some references. The authors were supported in part by DGAPA-UNAM grant IN121298 and by CONACyT Proy. Ref. J32754-E and 32272-E. U.N. would like to thank CONACyT for financial support.

## REFERENCES

- [1] P. T. Chrusciel, “No hair’ theorems: Folklore, conjectures, results,” *Contemp. Math.* **170**, 23 (1994). gr-qc/9402032.
- [2] R. Bartnik and J. Mckinnon, “Particle - Like Solutions Of The Einstein Yang-Mills Equations,” *Phys. Rev. Lett.* **61**, 141 (1988).
- [3] J. A. Smoller and A. G. Wasserman, “Existence of infinitely many smooth, static, global solutions of the Einstein / Yang-Mills equations,” *Commun. Math. Phys.* **151**, 303 (1993).
- [4] M. S. Volkov and D. V. Galtsov, “Nonabelian Einstein Yang-Mills Black Holes,” *JETP Lett.* **50**, 346 (1989) [*Pisma Zh. Eksp. Teor. Fiz.* **50**, 312 (1989)]. P. Bizon, “Colored Black Holes,” *Phys. Rev. Lett.* **64**, 2844 (1990).
- [5] J. A. Smoller, A. G. Wasserman and S. T. Yau, “Existence of black hole solutions for the Einstein / Yang-Mills equations,” *Commun. Math. Phys.* **154**, 377 (1993).
- [6] P. Bizon, “Gravitating solitons and hairy black holes,” *Acta Phys. Polon. B* **25**, 877 (1994) [gr-qc/9402016].
- [7] D. Nuñez, H. Quevedo and D. Sudarsky, “Black holes have no short hair,” *Phys. Rev. Lett.* **76**, 571 (1996) [gr-qc/9601020].
- [8] A. Ashtekar, C. Beetle, O. Dreyer, S. Fairhurst, B. Krishnan, J. Lewandowski and J. Wisniewski, “Isolated horizons and their applications,” *Phys. Rev. Lett.* **85**, 3564 (2000) [gr-qc/0006006].
- [9] A. Ashtekar, C. Beetle and S. Fairhurst, “Mechanics of Isolated Horizons,” *Class. Quant. Grav.* **17**, 253 (2000) [gr-qc/9907068]. A. Ashtekar and A. Corichi, “Laws governing isolated horizons: Inclusion of dilaton couplings,” *Class. Quant. Grav.* **17**, 1317 (2000) [gr-qc/9910068].
- [10] A. Ashtekar, S. Fairhurst and B. Krishnan, “Isolated horizons: Hamiltonian evolution and the first law,” *Phys. Rev. D* **62**, 104025 (2000) [gr-qc/0005083].
- [11] A. Corichi, U. Nucamendi and D. Sudarsky, “Einstein-Yang-Mills isolated horizons: Phase space, mechanics, hair and conjectures,” *Phys. Rev. D* **62**, 044046 (2000) [gr-qc/0002078].  
A. Corichi and D. Sudarsky, “Mass of colored black holes,” *Phys. Rev. D* **61**, 101501 (2000) [gr-qc/9912032].
- [12] A. Ashtekar, A. Corichi and D. Sudarsky, “Hairy black holes, horizon mass and solitons,” *Class. Quant. Grav.* **18**, 919 (2001) [gr-qc/0011081].
- [13] B. Kleihaus and J. Kunz, “Non-Abelian black holes with magnetic dipole hair,” *Phys. Lett. B* **494**, 130 (2000) [hep-th/0008034].
- [14] I. Racz and R. M. Wald, “Global extensions of space-times describing asymptotic final states of black holes,” gr-qc/9507055. I. Racz and R. M. Wald, “Extension of space-times with Killing horizon,” *Class. Quant. Grav.* **9**, 2643 (1992).
- [15] J. D. Bekenstein, “Nonexistence Of Baryon Number For Static Black Holes,” *Phys. Rev. D* **5**, 1239 (1972).
- [16] P. Breitenlohner, P. Forgacs and D. Maison, “On Static spherically symmetric solutions of the Einstein Yang-Mills equations,” *Commun. Math. Phys.* **163**, 141 (1994).